Andrade, Omori, and time-to-failure laws from thermal noise in material rupture

A. Saichev[1,2] and D. Sornette[2,3,4,*]

[1]*Mathematical Department, Nizhny Novgorod State University, Gagarin prospekt 23, Nizhny Novgorod 603950, Russia*

[2]*Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90095, USA*

[3]*Department of Earth and Space Sciences, University of California, Los Angeles, California 90095, USA*

[4]*Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France*

(Received 20 November 2003; revised manuscript received 17 August 2004; published 14 January 2005)

Using a simple mean-field rupture model with quenched disorder in the presence of thermal fluctuations introduced by S. Ciliberto *et al.*, we provide an analytical theory of three ubiquitous empirical observations obtained in creep (constant applied stress) experiments: the initial Andrade-like and Omori-like $1/t$ decay of the rate of deformation and of fiber ruptures and the $1/(t_c-t)$ critical time-to-failure behavior of acoustic emissions just prior to the macroscopic rupture. The lifetime of the material is controlled by a thermally activated Arrhenius nucleation process, describing the crossover between these two regimes, as shown by S. Ciliberto *et al.* Thus tiny thermal fluctuations may actually play an essential role in macroscopic deformation and rupture processes at room temperature. We also discover a reentrant dependence of the lifetime as a function of the amount of quenched disorder.

DOI: 10.1103/PhysRevE.71.016608 PACS number(s): 62.20.Mk, 05.70.Ln, 61.43.−j

Rupture in random media can be viewed as a kind of critical phenomenon [1,2], with proposed applications, in particular, to fiber composites and earthquakes. This field has attracted the attention of physicists due to the existence of power laws and fractals or multifractality expressing the self-organization of the rupture process. Constant stress (so-called "creep") experiments constitute a standard testing procedure in material sciences which exhibits a wealth of such "critical" behaviors. The typical response to the sudden application of a constant stress is that the strain rate first jumps rapidly to high values followed by slow universal power law decays, called the Andrade law [3], characterizing the "primary creep regime." Then, after this long decay followed by an approximately constant plateau (secondary regime) whose durations may vary within extraordinary large bounds (see below), the strain rate rebounds and accelerates (while the applied stress remains constant) by following a power law acceleration resulting in a finite-time singularity (the rupture of the sample): this is the tertiary creep regime. These regimes of decelerating followed by accelerating rates and the lifetime of the structure are the result of a subtle interplay between the preexisting microheterogeneity of the material and the self-organized evolving deformation and damage due to dislocation motion and/or microcracking. Previous pioneering models have suggested that primary and tertiary creeps are not independent [4].

Here, we propose a simple mechanism that provides an explanation of all these observations in a unified way. It is based on the recent proposal [5,6] that thermal noise is strongly amplified by quenched heterogeneities. Based on the analysis of a simple fiber bundle rupture model, Refs. [5,6] showed that the average lifetime of the fiber bundle takes an Arrhenius form with an effective temperature renormalized from the bare temperature T to a value strongly amplified by the presence of the frozen disorder in the rupture thresholds $f_c(i)$, in agreement with experiments and numerical simulations. This result suggests that the usual assumption of neglecting the role of thermal fluctuations in material rupture processes at room temperature may actually be incorrect (see [7] for early discussions): due to frozen heterogeneities, tiny thermal fluctuations can be amplified many times, thus actually controlling the time-dependent aspects of failure.

Since rupture involves a large range of scales, we follow the modeling strategy of critical phenomena (as well as material sciences) and use a coarse-grained model describing the mechanism of creep, damage and precursory rupture by averaging over the microscopic degrees of freedom to retain only two ingredients: (i) stress load transfer and (ii) thermal activation of the rupture of a coarse-grained element. The corresponding democratic fiber-bundle model (DFBM) with thermal noise [5,6] can be seen as a mean field treatment of rupture. A macroscopic constant load $F=Nf_0$ is applied at time $t=0$ to an initially undamaged system made of a very large number N of parallel elastic fibers (the results derived below are obtained in the thermodynamic limit $N\to\infty$). At all times, F is shared democratically among all $[1-\Phi(t)]N$ surviving fibers, where $\Phi(t)$ is the fraction of broken fibers at time t. The externally applied force per surviving fiber is thus $f_a=f_0/[1-\Phi(t)]$. The strength of each fiber i is characterized by a critical value $f_c(i)$ drawn for a distribution $P_d(f)$, centered on the mean equal to 1 and with variance T_d. Putting the mean strength to 1 sets the force scale. The heterogeneous strengths are given characteristic of the fibers and correspond to a quenched disorder, which is "read" in a certain organized way as the rupture develops. Microscopic thermal fluctuations are taken into account by assuming that a fiber with load f_a and threshold $f_c(i)>f_a$ has a nonzero probability $G(f_c(i)-f_a)$ to rupture per unit time governed by the rate with which a thermal fluctuation can activate a microscopic force $\Delta f_i \geqslant f_c(i)-f_a$ to pass the rupture threshold $f_c(i)$:

*Electronic address: sornette@moho.ess.ucla.edu

$$G(f_c(i)-f_a)=\frac{\gamma}{2}\,\mathrm{erfc}\left(\frac{f_c(i)-f_a}{\sqrt{2T}}\right), \tag{1}$$

where $\mathrm{erfc}(x)$ is the complementary error function, T is the variance of the thermal force fluctuations Δf_i, and γ is a microscopic constant rate fixing the time scale of the thermal activation process. This expression amounts to introducing a zero-mean normal distribution of thermal fluctuation forces Δf_i with variance T and with correlation time proportional to $1/\gamma$.

We first follow [6] and introduce the distribution $Q(f,t)$ of the rupture thresholds of the unbroken fibers at time t. Obviously, $Q(f,t)=0$ for $f<f_a$, since all these fibers are already broken. $Q(f,t)$ can be approximated with a very high accuracy in the limit $N\to\infty$ by the initial distribution $P_d(f)$ of rupture strengths truncated at a lower value $f_s(t)$,

$$Q(f,t)=P_d(f) \quad \text{for } f>f_s(t) \tag{2}$$

and 0 otherwise, where $f_s(t)$ is determined by the self-consistent equation

$$\Phi(t)=\int_{-\infty}^{f_s(t)} df\, P_d(f) \tag{3}$$

expressing that all fibers whose strengths are below $f_s(t)$ have failed at some time before t. This approximation for $Q(f,t)$ with Eq. (3) has been checked by extensive numerical simulations in [5] and amounts to viewing the time-dependent rupture as a "front" propagating and "eating" the distribution $P_d(f)$ from the weakest towards the strongest fibers. We also have by definition

$$\Phi(t)=1-\int_{-\infty}^{+\infty} df\, Q(f,t). \tag{4}$$

Taking the time derivative of $\Phi(t)$ and replacing $\dot{Q}(f,t)$ by $-Q(f,t)G(f-f_a)$ expressing that the rate of breaking is controlled by the thermally activated rupture process acting on each fiber independently, we get

$$\dot{\Phi}=\int_{-\infty}^{+\infty} df\, Q(f,t)G(f-f_a). \tag{5}$$

Putting Eq. (2) in this equation and taking for $P_d(f)$ a normal distribution centered on 1 with variance T_d as in [5,6] yields

$$\dot{\Phi}=\frac{\gamma}{2}\int_{f_s}^{\infty}\frac{1}{\sqrt{2\pi T_d}}\exp\left[-\frac{(1-f)^2}{2T_d}\right]\mathrm{erfc}\left(\frac{f-f_a}{\sqrt{2T}}\right)df. \tag{6}$$

Making explicit $P_d(f)$ in Eq. (3) gives

$$\Phi=\frac{1}{2}\left[\mathrm{erf}\left(\frac{f_s-1}{\sqrt{2T_d}}\right)+1\right], \quad f_s=1+\sqrt{2T_d}\,\mathrm{irf}(2\Phi-1), \tag{7}$$

where $y=\mathrm{irf}(z)$ is the inverse function to the error function $z=\mathrm{erf}(y)$. Putting f_s in Eq. (6) gives

$$\dot{\Phi}=R(\Phi)\equiv\frac{\gamma}{2}\int_{\Phi}^{1}\mathrm{erfc}[L(\Phi,z)]dz, \tag{8}$$

with

$$L(\Phi,z)=(1/\sqrt{2T})\left(1-\frac{f_0}{1-\Phi}\right)+\mu\,\mathrm{irf}(2z-1) \tag{9}$$

and

$$\mu=\sqrt{T_d/T}. \tag{10}$$

This equation is valid as long as the approximation (2) holds (see below). The solution of Eq. (8) provides in principle all the information on the fraction $\Phi(t)$ of broken fibers.

The parameter μ quantifies the relative importance of the thermal fluctuations compared with the quenched heterogeneities. The relevant regime for applications to macroscopic ruptures at room temperature is $\mu>1$ and often $\mu\gg1$, that is, thermal fluctuations are tiny contributions to the applied macroscopic mechanical forces. Indeed, assuming that the energy barrier to rupture a fiber corresponds to the Griffith energy $\approx gc^2$ necessary for nucleating a crack of half-length c in the solid with surface energy g, we obtain $\mu\approx1.5$–4×10^3 for $c=1$ micron and $\mu\approx1.5$–4 for $c=1$ nanometer, using $g=10$–50 erg/cm^2 for most solids. Thus, even for the smallest microcracks, thermal fluctuations are very small in relative value.

It turns out that this regime $\mu\gtrsim1$ allows for a very convenient approximation of $R(\Phi)$ obtained by linearizing $L(\Phi,z)$ with respect to z. Then, the integral over z in Eq. (8) can be calculated explicitly to yield

$$\dot{\Phi}=R(\Phi)=\frac{\gamma T}{4\pi\mu D(\Phi)U(\Phi)}e^{-U(\Phi)/T}, \tag{11}$$

where

$$U(\Phi)=TL^2(\Phi,\Phi)=\frac{1}{2}[f_s(\Phi)-f_a(\Phi)]^2 \tag{12}$$

and

$$D(\Phi)=(1/\sqrt{2\pi T_d})df_s(\Phi)/d\Phi=\exp\{\mathrm{irf}^2(2\Phi-1)\}. \tag{13}$$

$U(\Phi)$ in Eq. (12) has a clear physical interpretation as the energy barrier between the actual force

$$f_a(\Phi)=f_0/[1-\Phi(t)] \tag{14}$$

and the force front $f_s(\Phi)$ (7) of the distribution $Q(f,t)$ in Eq. (2). As the temperature goes to zero, the rupture rate $G(f-f_a)$ goes to zero for $f>f_a$ and thus $f_s\to f_a$, as it should. For a very small temperature, f_s adjusts itself dynamically in a self-consistent way slightly above f_a by the influence of the tiny thermal fluctuations which are just capable of passing over the effective potential barrier $U(\Phi)$. Equation (11) is valid as long as $U(\Phi)\gg T$, which implies $\Phi<\Phi_c$, where Φ_c is such that the force f_a per surviving fiber reaches the average strength 1: $f_a(\Phi_c)=1$ yielding $\Phi_c=1-f_0$. Such fractions $\Phi\to\Phi_c$ correspond to the ultimate regime of explosive failure. We have checked by direct numerical calculations that Eq. (11) provides an exceedingly precise approximation of Eq. (8) as long as $\mu\gtrsim1$ and not too close to Φ_c (in practice to within a few percent).

To go further, we need to distinguish between two regimes, $\Phi<\Phi^*$ and $\Phi>\Phi^*$, where Φ^*, solution of $dR(\Phi)/d\Phi=0$, corresponds to the minimum failure rate. For

$\mu \gtrsim 1$, Φ^* is actually independent of the temperature T and is the root of the equation

$$D(\Phi^*)(1-\Phi^*)^2 = \alpha, \quad \alpha = \frac{f_0}{\sqrt{2\pi T_d}}, \tag{15}$$

where α is an important physical parameter quantifying the strength of the disorder relative to the applied force. The explicit approximate solution of Eq. (15) is

$$\Phi^*(\alpha) = \begin{cases} \dfrac{20-\pi-4(4-\pi)\alpha}{24-2\pi+8(4+\pi)\alpha}, & \alpha < 3/2, \\ \dfrac{1}{2}\,\mathrm{erfc}(\sqrt{\ln \alpha})\dfrac{\alpha\sqrt{\pi \ln \alpha}}{1+\alpha\sqrt{\pi \ln \alpha}}, & \alpha > 3/2. \end{cases} \tag{16}$$

For example, this gives $\Phi^*(\alpha=2)=0.089$ compared with the exact value 0.092.

It follows from Eq. (11) that the time to reach some Φ is given by

$$\gamma T t = 4\pi\mu \int_0^{\Phi} D(z)U(z)e^{U(z)/T}dz. \tag{17}$$

For $0<\Phi<\Phi^*(\alpha)$, due to the exponential factor $e^{U/T}$, the main contribution to the last integral comes from a small neighborhood of the upper integration limit. This yields

$$\gamma t \simeq 4\pi\mu \frac{D(\Phi)U(\Phi)}{A(\Phi)} e^{U(\Phi)/T}, \quad A(\Phi) = \frac{dU(\Phi)}{d\Phi}, \tag{18}$$

for $\Phi<\Phi^*$. This approximation is correct under the assumption that $e^{U(\Phi)/T}$ is rapidly increasing with Φ, i.e., if $|A(\Phi)| \gg T$. The absolute value $|\cdots|$ stresses that this condition applies also for $\Phi>\Phi^*(\alpha)$. The same reasoning in this case gives a similar approximation

$$\gamma(t_c - t) \simeq 4\pi\mu \frac{D(\Phi)U(\Phi)}{|A(\Phi)|} e^{U(\Phi)/T}, \quad \Phi > \Phi^*, \tag{19}$$

where $t_c - t$ is the time to complete rupture. The condition $|A(\Phi)| \gg T$ shows that both relations (18) and (19) do not work in the vicinity of the minimum rate of fiber failures given by the solution of Eq. (15), for which $A(\Phi^*)=0$. For $0<\Phi<\Phi^*$, combining Eqs. (11) and (18), we obtain

$$\dot{U} = T/t \quad \text{for } t < t^*, \tag{20}$$

where t^* is defined by $\Phi(t^*)=\Phi^*$. This gives

$$U(\Phi(t)) = T \ln \gamma t \quad \text{for } 1 \ll t < t^*. \tag{21}$$

The first inequality simply means that the thermal fluctuations have had time to contribute several independent jolts. The constant of integration gives the $\ln \gamma$ contribution determined from matching with the initial stage. Replacing the left-hand side of Eq. (12) by $U[\Phi(t)] = T \ln \gamma t$ and putting $f_a \simeq f_0$ (for Φ small) gives, in view of Eq. (11), the fraction rate

$$\dot{\Phi} \simeq \frac{1}{4\pi\mu t \ln \gamma t} \exp\left[-\frac{1}{2T_d}(1 - f_0 - \sqrt{2T \ln \gamma t})^2 \right]. \tag{22}$$

Expression (22) was obtained numerically in Ref. [6] and our analysis extends this previous work by providing a direct analytical derivation. This expression shows that the failure rate $\dot{\Phi}$ of fibers decreases after application of the load proportionally to $1/t$, up to logarithm corrections. This $1/t$ decay lasts as long as Φ remains smaller than Φ^*. This $1/t$ law is known in seismology as the Omori law [8]. It is also ubiquitous in creep experiments with exponents that are often close to or smaller than our prediction 1. For intermediate times such that $\gamma t < e^{1/2T}$,

$$\dot{\Phi} \sim \frac{1}{t \ln \gamma t} e^{(1-f_0)\sqrt{2T \ln \gamma t}/T_d}, \tag{23}$$

which gives an apparent exponent $\sim 1/t^p$ with $p<1$. For $\ln \gamma t \gg (1-f_0)^2/2T$, $p \to 1+(T/T_d)$ which is close to but slightly larger than 1. Numerical simulations confirm these predictions accurately. See, for instance, Fig. 2 of [6] which our theory explains quantitatively. Exact numerical integration and our analytical approximation coincide everywhere, excluding a time interval corresponding to a very small vicinity of the stationary point Φ^*. Note that Andrade's law [3] also derives from the deformation rate being proportional to

$$df_a(t)/dt = f_0\dot{\Phi}/[1-\Phi(t)]^2 \propto \dot{\Phi} \tag{24}$$

as $\Phi(t)$ varies much more slowly than $\dot{\Phi}$.

Let us now turn to the description of the second regime $\Phi(t)>\Phi^*$, relevant to obtain the failure rate up to global failure. Combining Eqs. (11) and (19), we obtain the expression

$$U(\Phi(t)) = T \ln[\gamma(t_c - t)] \quad \text{for } \Phi^* < \Phi < \Phi_c. \tag{25}$$

The regime $\Phi^* < \Phi(t) < \Phi_c$ is strongly influenced by thermal fluctuations, so that the disorder term can be neglected to obtain, in view of the just quoted expression for $U(\Phi(t))$ and of Eq. (12),

$$\Phi_c - \Phi(t) = f_0 \frac{\sqrt{2T \ln \gamma(t_c - t)}}{1 - \sqrt{2T \ln \gamma(t_c - t)}}. \tag{26}$$

Differentiating both sides of this expression with respect to t yields the failure rate

$$\dot{\Phi}(t) = C(t)/(t_c - t), \tag{27}$$

where $C(t) = f_0 T/[c(1-c)^2]$ with $c = \sqrt{2T \ln[\gamma(t_c - t)]}$. This second important result was also obtained as Eq. (B11) in Ref. [5] by a different method. Expression (27) shows that, for $\Phi>\Phi^*$, the failure rate accelerates towards the finite-time singularity (27). The underlying physics of thermally activated failures of heterogeneous elements provides a novel mechanism for the ubiquitous time-to-failure regime observed in heterogeneous material [9]. Strong quenched heterogeneity has been shown to play an essential role in controlling the critical nature of the rupture process [2] and in the existence of a time-to-failure power law such as Eq. (27). Here, we confirm that the heterogeneity is essential to

renormalize the thermal fluctuations [5,6]. While the philosophy is similar, the mechanism is different. As for the Omori law, the logarithmic corrections in Eq. (27) may give an apparent exponent of the power law, slightly smaller than 1, as observed in experiments. Our numerical tests show that expression (27) provides an approximation which coincides almost everywhere with the exact solution inside the interval $t^* < t < t_c$.

There is a simple physical interpretation of the transition between the two aforementioned rate behaviors (22) and (27). To explain the first (rate decaying) regime, consider the degenerate case of spontaneous fracture ($f_0=0, \Phi^*=1$) for which $U(\Phi)=f_s^2/2$. As time increases, f_s grows, the remaining fibers are stronger and the failure rate decays together with the rate of change of the energy barrier. The second regime can be qualitatively understood by taking the limit of zero disorder ($T_d=0, \Phi^*=0$), leading to $U(\Phi)=(1-f_a)^2/2$. The force f_a per remaining fiber grows with time, the fibers break more and more easily and the failure rate grows to give the fracture in finite time. In the intermediate case $0<\Phi^*<\Phi_c$, due to the competition between the growth of f_s and f_a, the two regimes coexist. At early times, the growth of f_s dominates giving the Omori and Andrade laws, followed by the growth of f_a in the second regime $\Phi>\Phi^*$ giving the power law finite-time singularity.

Last, we turn to the behavior for $\Phi \approx \Phi^*$, which turns out to provide the dominant contribution for the total time for rupture, as shown in [5,6]. Indeed, the fiber bundle spends most of its time in the vicinity of the stationary point Φ^*, corresponding to the minimum failure rate. In this case, U can be expanded as

$$U(\Phi) = U(\Phi^*) - B(\Phi^*)(\Phi - \Phi^*)^2, \tag{28}$$

with $B(\Phi)=-\frac{1}{2}[d^2U(\Phi)/d\Phi^2]$, and Eq. (11) becomes

$$\dot{\Phi} \simeq R(\Phi^*)\exp\left[-\frac{B(\Phi^*)}{T}(\Phi - \Phi^*)^2\right]. \tag{29}$$

The solution of this equation is

$$\mathrm{erfi}\left(\sqrt{\frac{B(\Phi^*)}{T}}(\Phi - \Phi^*)\right) = 2\sqrt{\frac{B(\Phi^*)}{\pi T}}R(\Phi^*)(t - t^*), \tag{30}$$

where $\mathrm{erfi}(z)=(1/i)\mathrm{erf}(iz)$ is the imaginary error function. Using its asymptotics $\mathrm{erfi}(z)\sim(1/\sqrt{\pi}z)e^{z^2}$ for large z together with Eq. (30), Eq. (29) becomes

$$d\Psi^2/dt \simeq \frac{T}{B(\Phi^*)(t - t^*)}, \quad \text{with } \Psi = \Phi - \Phi^*, \tag{31}$$

whose solution yields the fracture rate

$$\dot{\Phi} \simeq \sqrt{T}\,\mathrm{sgn}(t - t^*)/2|t - t^*|\sqrt{B(\Phi^*)\ln(\gamma|t - t^*|)}, \tag{32}$$

for $\gamma|t-t^*| \gg 1$. Expression (29) allows us additionally to calculate the total lifetime t_c of the fiber bundle:

$$\gamma t_c = [1/R(\Phi^*)]\int_{-\infty}^{\infty} \exp\{-[B(\Phi^*)/T](\Phi - \Phi^*)^2\}d\Phi. \tag{33}$$

The calculation of this integral with the use of Eq. (11) gives

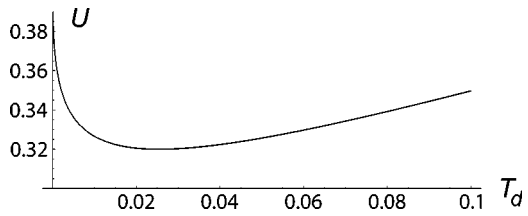

FIG. 1. Effective dimensionless barrier energy $U(\Phi^*)$ as a function of the dimensionless disorder strength T_d, for $\Phi^*=1/2$, corresponding to $T_d^*=(8/\pi)f_0^2 \simeq 0.025$ for $f_0=0.1$.

$$\gamma t_c \simeq 4\pi\sqrt{\pi}\frac{D(\Phi^*)U(\Phi^*)}{\sqrt{B(\Phi^*)}}\frac{\sqrt{T_d}}{T}\exp\left[-\frac{U(\Phi^*)}{T}\right]. \tag{34}$$

This expression, together with Eq. (16), recovers the main result of [5,6], while improving on the prefactors to the main Arrhenius-type dependence.

Using Eqs. (12) and (15), $U(\Phi^*)$ can be written explicitly

$$U(\Phi^*) = \left[\frac{\Phi_c - \Phi^*}{1 - \Phi^*} \pm \sqrt{\ln\left(\frac{\alpha}{(1 - \Phi^*)^2}\right)}\right]^2, \tag{35}$$

where the sign + (−) corresponds to the case $\Phi^*>1/2$ ($\Phi^*<1/2$). As shown in Fig. 1, $U(\Phi^*)$ is a nonmonotone function of T_d. Due to the aforementioned competition between quenched disorder and the growth of the actual force f_a, $U(\Phi^*)$ decreases as long as $T_d<T_d^*$ and then increases with increasing T_d beyond T_d^*. The first regime $T_d<T_d^*$ corresponds to the effect discovered in Refs. [5,6] and mentioned above of the renormalization of thermal fluctuations by quenched disorder, and consequently of decreasing strength by increasing the disorder. Since a larger $U(\Phi^*)$ corresponds to a large lifetime through Eq. (34), we uncover the effect of a strengthening of the fiber system by increasing the disorder beyond a certain threshold. All our formulas have been checked by direct numerical integration with excellent agreements. We expect that extensions of the DFBM to nonmean field power law interactions [10] will not change our results qualitatively but may modify the Omori's and time-to-failure exponents.

In conclusion, we have revisited the coarse-grained thermally activated DFBM model introduced in Refs. [5,6]. This model captures the collective nature of many small-scale thermally-activated processes (dislocation motion, cavities, microcracks) via the cascade of thermally-activated load transfers reorganizing the stress field described above. Complementing previous analyses [5,6], we have presented a synthetic analytical derivation of the Andrade law of the primary creep regime, which is due to the thermally activated stress transfer on the most susceptible elements (weakest fibers). We have also recovered previous analytical results of the tertiary power law regime, which is due to the cooperative cascade involving a finite fraction of the whole system. In addition, we predict a reentrant dependence of the lifetime as a function of the amount of quenched disorder.

We acknowledge useful exchanges with S. Ciliberto and A. Politi.

[1] *Statistical Models for the Fracture of Disordered Media*, edited by H. J. Herrmann and S. Roux (Elsevier, Amsterdam, 1990).

[2] J. V. Andersen, D. Sornette, and K.-T. Leung, Phys. Rev. Lett. **78**, 2140 (1997), and references therein.

[3] E. N. da C. Andrade, Proc. R. Soc. London, Ser. A **84**, 1 (1910); **90**, 329 (1914).

[4] A. Dlouhy *et al.*, Acta Metall. Mater. **43**, 535 (1995); G. S. Daehn, Acta Mater. **49**, 2017 (2001).

[5] S. Ciliberto, A. Guarino, and R. Scorretti, Physica D **158**, 83 (2001).

[6] A. Politi, S. Ciliberto, and R. Scorretti, Phys. Rev. E **66**, 026107 (2002).

[7] R. I. B. Selinger, Z. Wang, and W. M. Gelbart, Phys. Rev. A **43**, 4396 (1991), and references therein; A. Buchel and J. P. Sethna, Phys. Rev. Lett. **77**, 1520 (1996).

[8] T. Utsu, Y. Ogata, and S. Matsu'ura, J. Phys. Earth **43**, 1 (1995).

[9] J.-C. Anifrani *et al.*, J. Phys. I **5**, 631 (1995); A. Garcimartin *et al.*, Phys. Rev. Lett. **79**, 3202 (1997); see also S. G. Sammis and D. Sornette, Proc. Natl. Acad. Sci. U.S.A. **99**, S1, 2501 (2002).

[10] F. Kun, Y. Moreno, R. C. Hidalgo, and H. J. Herrmann, Europhys. Lett. **63**, 347 (2003).